

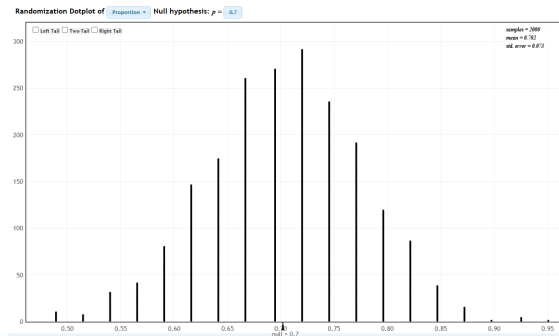
One-Sample Hypothesis Testing

Part 2 - the Z-test

Ryan Miller

Introduction

So far, we've relied upon *simulations* to determine the *null distribution* and *p*-value for our hypothesis tests.



You might have noticed that these simulations often produce a *bell-shaped* distribution.

Normal Distributions

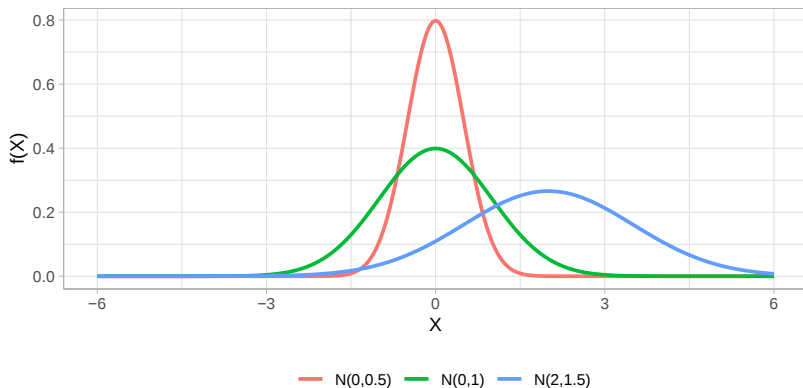
This shape is no coincidence, but rather a consequence of **Central Limit theorem** (CLT). But before we discuss CLT, we'll need to cover a few details about the **Normal distribution**:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- ▶ μ is the center (mean) of the distribution
- ▶ σ is the standard deviation of the distribution
- ▶ We use the shorthand $N(\mu, \sigma)$ to express a Normal distribution
 - ▶ For example, $N(3, 1)$ is a curve centered at 3 with a standard deviation of 1
- ▶ You don't need to know the formula for the Normal curve, but you should know that it depends on μ and σ

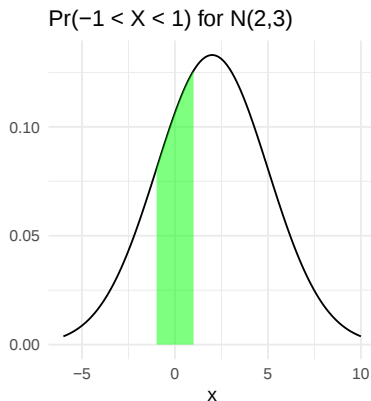
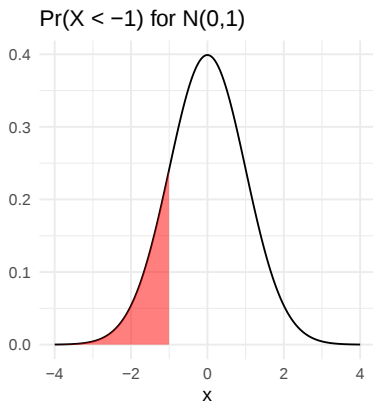
Normal Distributions

Below are three different Normal Distributions displayed on the same x-axis:



Probability and the Normal Distribution

The Normal curve is a **probability distribution**, meaning the area under the curve can be used to model/calculate the probabilities of certain events:



Z-Scores

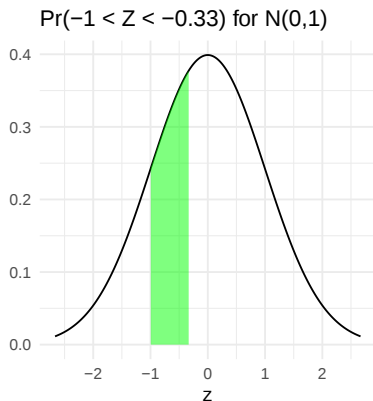
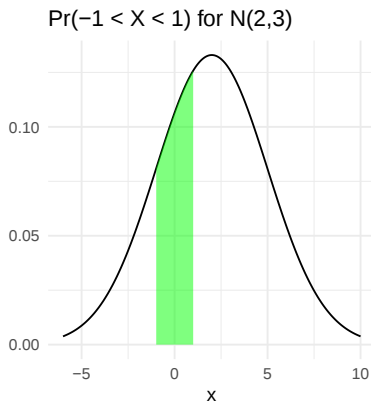
Because it is inconvenient to work with a distribution that uses a different scale in each new analysis statisticians frequently use a standardization approach known as the **Z-score transformation**:

$$Z = \frac{\text{Observed value} - \text{Expected Value}}{\text{Standard Deviation}}$$

- ▶ For example, an ICU patient in the data from our previous lab had a systolic blood pressure of 162.
 - ▶ The mean systolic blood pressure of the entire sample was 132.28, and the sample standard deviation was 32.95
- ▶ So, this individual's Z-score is $Z = \frac{162 - 132.28}{32.95} = 0.90$
 - ▶ Meaning they are almost 1 standard deviation above the sample average

Z-Scores and Probability

Z-score transformations allow us to use the $N(0,1)$ curve as a probability model for any scenario, regardless of the measurement units:



Central Limit Theorem

Mathematically, **Central Limit theorem** (CLT) states:

$$\lim_{n \rightarrow \infty} \sqrt{n} \left(\frac{\bar{X} - \mu}{\sigma} \right) \rightarrow N(0, 1)$$

In more practical terms (using slightly informal notation), CLT suggests any random variable that is an average of sufficiently many independent observations will follow a Normal distribution with a predictable mean and standard deviation:

$$\bar{X} \approx N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

Central Limit Theorem

In hypothesis testing, we can make this even more general:

Sample Estimate $\sim N(\text{Expected value under } H_0, SE)$

$$\Rightarrow \frac{\text{Sample Estimate} - \text{Expected value under } H_0}{SE} \sim N(0,1)$$

Here SE is the **standard error** of the sample estimate. We won't get into how these are derived, but CLT gives us the following SE formulas:

- ▶ $SE = \sqrt{\frac{p \cdot (1-p)}{n}}$ for a single proportion
- ▶ $SE = \frac{\sigma}{\sqrt{n}}$ for a single mean (note that σ is the standard deviation of cases in the population)

The Z-test

Central Limit theorem allows for a standardized hypothesis testing approach whenever we are studying a sample average or a sample proportion (which is just an average of 0's and 1's)

1. Apply the Z-score transformation to the observed sample mean or proportion using the null hypothesis and SE derived from CLT
2. Compare the resulting Z-score to a $N(0,1)$ distribution to find the p -value by looking at the area corresponding to Z-scores at least as extreme as the one from the sample data

Z-test Example

In our infant toy choice example, $H_0 : p = 0.5$ and we observed $\hat{p} = 14/16 = 0.875$, or 14 of 16 infants choosing the “helper”. Carry out a one-sample Z-test by performing the following steps:

- 1) Identify the expected value and SE for the Z-transformation
- 2) Calculate the test statistic (Z-value)
- 3) Use the “Normal” menu of StatKey (under Theoretical Distributions) to calculate the one-sided p -value

Z-test Example (solution)

- 1) Under H_0 , CLT implies $SE = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.5(1-0.5)}{16}} = 0.125$
- 2) Then, $Z = \frac{\hat{p}-p}{SE} = \frac{0.875-0.5}{0.125} = 3$
- 3) Using a $N(0,1)$ distribution, $Pr(Z \geq 3) = 0.0013$, which is the one-sided p -value. The two-sided p -value would be 0.0026 due to the symmetry of the Normal distribution

Conclusion

- ▶ The Z-test uses a Normal distribution as probability model for the null distribution
 - ▶ This model may or may not accurately reflect the actual null distribution
- ▶ The Z-test is generally considered appropriate for one-sample categorical data when at least 10 instances of each outcome were observed
 - ▶ $n\hat{p} \geq 10$ and $n(1 - \hat{p}) \geq 10$
- ▶ If the above conditions are not met, we should seek out an alternative approach (we'll discuss non-parametric tests later on)