Confidence Intervals

Part 2 - Normal Approximations

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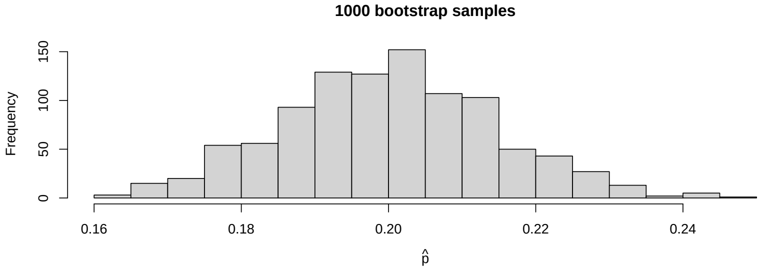
Normal Distributions

We've now seen several *bootstrap distributions* and you may have noticed they tend to be "bell-shaped":



This is not a coincidence, it's backed up by statistical theory

Normal Distributions

Bootstrap distributions can be characterized by the curve:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- This curve defines the **Normal Distribution**
  - μ is the center (mean) of the distribution
  - σ is the standard deviation of the distribution
  - We use the shorthand $N(\mu, \sigma)$ to express a normal distribution, for example: N(3,1) is a curve centered at 3 with a standard deviation of 1
- You don't need to know the formula for the normal curve, though you should know that it depends on μ and σ

Normal Approximation

- When calculating a confidence interval estimate, we can use a *normal approximation* instead of bootstrapping
  - To do this, we need the distribution's mean and standard deviation (since any normal curve is entirely by μ and σ)
  - Thus, the approximation will be $N(\text{estimate}, \text{SE})$
    - We saw the bootstrap distribution was centered around the estimate from the original sample
    - We generated bootstrap samples and bootstrap statistics to find SE, but is there another way?

Central Limit Theorem

- The **Central Limit Theorem** (CLT), one of the most well-known results in statistics, provides a mathematical expression for the *SE* of many commonly used descriptive statistics
  - We'll first look at a CLT result for *one proportion*:

$$\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

In words, the sample proportion, $\hat{p}$, follows a normal distribution with a mean of p and standard deviation of $\sqrt{\frac{p(1-p)}{n}}$, thus providing a normal approximation of the sampling distribution

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Using the CLT (one proportion)

Central Limit theorem gives us:

$$\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

- Thus, $SE = \sqrt{\frac{p(1-p)}{n}}$ when estimating a *single proportion*
- We don't know p, but $\hat{p}$ is our *best estimate*, together these suggest the 95% confidence interval:

$$\hat{p} \pm 2 * \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

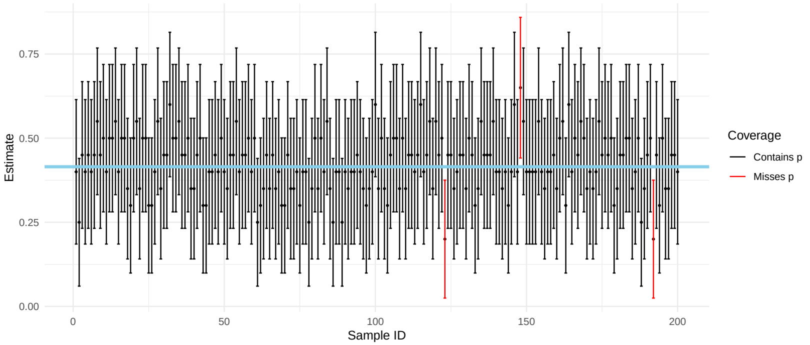
Confidence Interval Coverage

The phrase "95% confidence" describes the long-run success rate of *the procedure* used to calculate the interval. So let's apply the procedure from the previous slide to many random samples of size $n = 20$ from a population with $p = 0.415$:

Sample ID	Sample proportion	Calculation	95% CI
1	0.4	0.4 +/- 2* 0.11	(0.181,0.619)
2	0.25	0.25 +/- 2* 0.097	(0.056,0.444)
3	0.45	0.45 +/- 2* 0.111	(0.228,0.672)
4	0.4	0.4 +/- 2* 0.11	(0.181,0.619)
5	0.45	0.45 +/- 2* 0.111	(0.228,0.672)
6	0.4	0.4 +/- 2* 0.11	(0.181,0.619)

Confidence Interval Coverage

When we apply this procedure 200 times, only 3 intervals fail to capture the true p, suggesting the procedure is valid (but perhaps slightly conservative):



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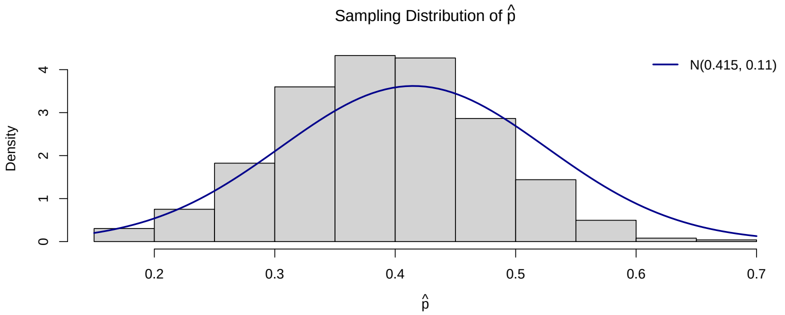
Confidence Interval Coverage

A long-run success rate that is slightly above 95% makes sense, as a normal approximation of the sampling distribution is decent but not perfect:



Practice

In a study conducted by Johns Hopkins University researchers investigated the survival of babies born prematurely. They searched their hospital's medical records and found 39 babies born at 25 weeks gestation (15 weeks early), 31 of these babies went on to survive at least 6 months. With your group:

1. Use a normal approximation to construct a 95% confidence interval estimate for the true proportion of babies born at 25 gestation that are expected to survive.
2. An article on Wikipedia suggests 70% of babies born at 25 weeks gestation survive. Is this claim consistent with the Johns Hopkins study?

Practice - Solution

1. $\hat{p} = 31/39 = 0.795$, using the normal approximation provided by CLT, $SE = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.795(1-0.795)}{39}} = 0.065$; this suggests the 95% CI:
$$0.795 \pm 2 * 0.065 = (0.668, 0.922)$$
2. Yes, 0.70 is contained in the 95% confidence interval, suggesting it is a plausible value of the population parameter.

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Sufficiently Large?

The normal approximation suggested by the Central Limit Theorem is only accurate when n is sufficiently large

- For a single proportion, "sufficiently large" also depends upon the value of p
- A common rule of thumb for whether this normal approximation of $\hat{p}$ is reasonable requires:

1. $n * p \geq 10$
2. $n * (1 - p) \geq 10$

If *either* of these conditions isn't met you should consider an alternative (our lab will introduce *exact binomial confidence intervals*)

Confidence Levels that aren't 95%

Confidence intervals have the form:

$$\text{Estimate} \pm c * SE$$

- Normal approximations allow us to achieve any confidence level via the choice of "c"
  - "c" is chosen as a cut-point from the **standard normal** distribution, which has a mean of 0 and standard deviation of 1
  - The "Theoretical Distribution" menu on StatKey helps us find the cut-point defining to the middle P% of the distribution (yielding a P% CI)

Conclusion

We've now seen how to use a Normal approximation to construct a confidence interval estimate for a *single proportion*.

- We can estimate p using an interval of the form:

$$\hat{p} \pm c * \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

This formula is only reliable when the sample is *sufficiently large*. Exact approaches should be used for small samples.